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Lagrangian Solution of Schwarzschild-like Metric for an Elliptical Object

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ABSTRACT

Lagrangian method applied as well as tensor method, for a linear transformed geodesic line element of Schwarzschild-like The Lagrangian method was applied for a linearly transformed geodesic line element of a Schwarzschild-like solution instead of the tensor method. The solution shows that it is not only valid for spherical objects but also it is more comprehensive for elliptical celestial objects. Two types of kinetic and potential energy are the basis of the calculation. Hamiltonian and Lagrangian equality show that the problem has no potential energy. With this transformed geodesic line element, we obtained a new coefficient for the meridional advance of an experimental particle in Schwarzschild spacetime in terms of period, eccentricity, and mean distance. This new perigee equation is not only valid for the Schwarzschild metric (for a spherical object), but also more accurate for the Schwarzschild-like metric (for elliptical objects).

Keywords: Schwarzschild-like solution; Lagrangian solution; Spherical object; Elliptical objects.

INTRODUCTION

Parallel displacement can be used to define a special class of curves called geodesics [1, 2]. The geodesic definition is the shortest line between two points on the curved manifold [3-5]. A geodesic may be classified into three different types [6, 7] based on its tangent vector. The tangent vector such as $\vec{V} = dx/d\lambda$, is, a curve that satisfies $\nabla\vec{V} = 0$. These three types are as time-like ($\vec{V}.\vec{V} < 0$), null ($\vec{V}.\vec{V} = 0$), or space-like ($\vec{V}.\vec{V} > 0$) [8]. The geodesic equation can describe the motion of a particle in a space [9] which is given by the following equation.

$$\frac{dx^{\alpha}}{ds^2} + \Gamma^{\alpha}_{\mu\nu}\frac{dx^{\mu}}{ds}\frac{dx^{\nu}}{ds} = 0 \tag{1}$$

This equation is derived from the tangent vector to the curve λ as it is parallel propagated along λ [10]. The equation which is governing the geodesic in a space-time [11] with the line element

$$ds^2 = g_{ij}dx^i dx^j \tag{2}$$

also can be derived from the Euler-Lagrange equation[12],

$$\frac{d}{ds}\left(\frac{\partial\mathcal{L}}{\partial\dot{x}^j}\right) - \frac{\partial L}{\partial x^j} = 0 \tag{3}$$

Here $\dot{x}^j$ is defined as $dx^j\lambda(s)/ds$ and $\frac{\partial\mathcal{L}}{\partial\dot{x}^j} = 2g_{ij}\,\partial\dot{x}^i$ and $\partial\mathcal{L}/\partial x^j = g_{ik,j}\,\dot{x}^i\dot{x}^k$ or in more familiar form as,

$$2\mathcal{L} = g_{ij}\frac{dx^i}{d\tau}\frac{dx^j}{d\tau} \quad (4)$$

With the exception of the singular case in which the function $\mathcal{L}$ is depends on some or all the $\dot{x}^j$ the differentiation with respect to S brings in all the second derivatives $\ddot{x}^j$ [13].
Here, τ is defined as affine parameter along the geodesic [10, 14]. For time-like geodesics [15, 16] parameter of τ may be identified with the proper time S of the particle describing the geodesic. The concept of proper time [17] was introduced by Hermann Minkowski in 1908 [18, 19] and it is the time measured by a clock of standard construction travelling with the particle [20]. The first aim of the present paper is to solve a geodesic line element like Schwarzschild metric which in this paper we call Schwarzschild-like metric [21, 22]. This metric not only can describe for spherical object but it is able to consider for elliptical celestial object too. Here the Lagrangian method has been used. The result of this method in compare with tensor or Riemannian geometry method is same (see [10]).
In general, the solution of geodesic metric with using tensor method for spherical object may be is not very hard but it is very tedious in calculations of Christoffel tensor. Because each combination of indexes i, j, and k should be to calculate with 12 derivatives of the metric tensors. The tensor method for non-spherical coordinate systems is really more difficult and very tedious too. So, we would like to consider another more efficient way to get these results.

DERIVATION OF ELLIPTICAL METRIC BY LAGRANGIAN

The expression of

$$ds^2 = c^2dt^2 - \left(\frac{r^2 + a^2cos^2\theta}{r^2 + a^2}\right)dr^2$$
$$-(r^2 + a^2cos^2\theta)d\theta^2 - (r^2 + a^2)sin^2\theta d\varphi^2 \quad (5)$$

is the Galilean metric of

$$ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2 \quad (6)$$

written in especially oblate spheroidal coordinate system which the transformation to Cartesian coordinate system is accomplished with the following relation.

$$x = (r + a)^{1/2}sin\theta\, cos\varphi$$
$$y = (r + a)^{1/2}sin\theta\, cos\varphi$$
$$z = rcos\theta \quad (7)$$

The surfaces $r = const$ are oblate ellipsoid.

The line element (5) (for $a \neq 0$), describe the gravitational field of elliptical objects such as star and or planet and for $(a = 0)$ is also valid for spherical object [21]. In this coordinate system, a is a constant in the, $x - y$ surface. This line element in the presence of a mass point can takes the following form.

$$ds^2 = e^{\nu}dt^2 - e^{\lambda}\left(\frac{r^2 + a^2cos^2\theta}{r^2 + a^2}\right)dr^2$$
$$-(r^2 + a^2cos^2\theta)d\theta^2 - (r^2 + a^2)sin^2\theta d\varphi^2 \quad (8)$$

Here the values of e^{ν} and e^{λ} are as coefficient and ν and λ are function of r and θ. The velocity of light c, here is supposed as unit $(c = 1)$. By applying tensor method and Christoffel symbols and some calculation, finally we get,

$$ds^2 = \left(1 - \frac{2M}{r}\right)dt^2 - \frac{1}{\left(1 - \frac{2M}{r}\right)}\frac{(r^2 + a^2cos^2\theta)}{(r^2 + a^2)}dr^2$$
$$-(r^2 + a^2cos^2\theta)d\theta^2 - (r^2 + a^2)sin^2\theta d\varphi^2 \quad (9)$$

Here, M and r, are the mass and radius of ellipsoidal object, respectively. The equation of (9) for $(a \neq 0)$, is describing the gravitational field of elliptical object with Schwarzschild-like metric and for $(a = 0)$ the result shows the general form of Schwarzschild metric for spherical object in shape.
The solution of this line element (9) with using Einstein's field equations and Christoffel symbols already have done see (Ref [21]). The alternative solution is working in exact ellipsoidal coordinate system (u, v, ϕ), but it is very tedious and with more mathematics [23].
The Lagrangian equation of this line element is,

$$2\mathcal{L} = \left[\left(1 - \frac{2M}{r}\right)\dot{t}^2 - \frac{1}{\left(1 - \frac{2M}{r}\right)}\frac{(r^2 + a^2cos\theta)}{(r^2 + a^2)}\dot{r}^2\right.$$
$$\left. - (r^2 - a^2cos^2\theta)\dot{\theta}^2 - (r^2a^2)sin^2\theta\dot{\varphi}^2\right] \quad (10)$$

We suppose the motion of particle is along to the $\theta = \pi/2$. Accordingly, $cos\theta = 0$, $sin\theta = 1$ and $d\theta/dt = \dot{\theta} = 0$ and therefore the above equation is,

$$\therefore \mathcal{L} = \frac{1}{2}\left[\left(1 - \frac{2M}{r}\right)\dot{t}^2 - \frac{1}{\left(1 - \frac{2M}{r}\right)}\frac{r^2}{(r^2 + a^2)}\dot{r}^2 - (r^2 - a^2)\dot{\varphi}^2\right] \quad (11)$$

Here the dot denotes differentiation with respect to τ. Physically these conditions are corresponding to requiring the motion of particle which take place in a plane. The corresponding canonical momenta are

$$P_t = \frac{\partial \mathcal{L}}{\partial \dot{t}} = (1 - \frac{2M}{r})\dot{t} \tag{12}$$

$$P_r = -\frac{\partial \mathcal{L}}{\partial \dot{r}} = \frac{1}{(1-\frac{2M}{r})}\frac{r^2}{(r^2+a^2)}\dot{r} \tag{13}$$

$$P_\theta = -\frac{\partial \mathcal{L}}{\partial \dot{\theta}} = (2a^2 sin\theta)\dot{\theta} = 0 \quad (\text{for } \theta = \frac{\pi}{2}) \tag{14}$$

$$P_\varphi = -\frac{\partial \mathcal{L}}{\partial \dot{\varphi}} = (r^2 + a^2)\dot{\varphi} \tag{15}$$

Therefore, the resulting Hamiltonian is

$$H = [P_t\dot{t} - P_r\dot{r} - P_\theta\dot{\theta} - P_\varphi\dot{\varphi}] - \mathcal{L} = \mathcal{L} \tag{16}$$

The equality of the Hamiltonian and Lagrangian signifies that there is no potential energy in the problem. The constancy of the Hamiltonian and of the Lagrangian follows from this fact that $H = \mathcal{L} = cte$. By rescaling the affine parameter τ, we can arrange that $2\mathcal{L}$ has the value $+1$ for time-like geodesics. For null geodesics $\mathcal{L}$, has the value of zero which is space-like geodesic.

$$P_t = \frac{\partial \mathcal{L}}{\partial \dot{t}} = \left(1 - \frac{2M}{r}\right)\frac{dt}{d\tau} = constant = k \tag{17}$$

$$P_\varphi = -\frac{\partial \mathcal{L}}{\partial \dot{\varphi}} = (r^2 + a^2)\frac{d\varphi}{d\tau} = cte = h \,(\text{for } \theta = \frac{\pi}{2}) \tag{18}$$

Here h is the angular momentum about an axis normal to the invariant plane and k is a constant. Using the equations (17) and (18), the Lagrangian equation is

$$2\mathcal{L} = \left[\frac{k^2}{\left(1 - \frac{2M}{r}\right)} - \frac{r^2}{\left(1 - \frac{2M}{r}\right)(r^2 + a^2)}\dot{r}^2 - \frac{h^2}{(r^2 - a^2)}\right] = +1 \; or \; 0 \tag{19}$$

The values of $+1$ and 0, are depending on whether we are considering time-like or null geodesics respectively. Here we are considering the time-like geodesics and therefore with substituting (18) in (19) the geodesic equations is

$$\frac{k^2}{\left(1 - \frac{2M}{r}\right)} - \frac{r^2}{\left(1 - \frac{2M}{r}\right)(r^2 + a^2)}\left(\frac{dr}{d\tau}\right)^2 - \frac{h^2}{(r^2 + a^2)} = 1 \tag{20}$$

Using $\frac{dr}{d\tau} = \frac{dr}{d\varphi}\frac{d\varphi}{d\tau} = \frac{h}{(r^2+a^2)}\frac{d\varphi}{d\tau}$, in (20) then we have,

$$\frac{k^2}{\left(1 - \frac{2M}{r}\right)} - \left[\frac{r^2h^2}{\left(1 - \frac{2M}{r}\right)(r^2 + a^2)^3}\right]\left(\frac{dr}{d\varphi}\right)^2 - \frac{h^2}{(r^2 + a^2)} - 1 = 0 \tag{21}$$

and therefore

$$\left(\frac{dr}{d\varphi}\right)^2 = \frac{(r^2 + a^2)^3K^2}{r^2h^2} - \frac{(r^2 + a^2)^2}{r^2} + \frac{2M(r^2 + a^2)^2}{r^3} - \frac{(r^2 + a^2)^3}{r^2h^2} + \frac{2M(r^2 + a^2)^3}{r^3h^2} \tag{22}$$

By substituting $u = 1/r$ in (22), and rearranging then the result is as

$$\left(\frac{dr}{d\varphi}\right)^2 + (1 + a^2u^2)^2u^2 = \left(\frac{K^2 - 1}{h^2}\right)(1 + a^2u^2)^3 + \left(\frac{2Mu}{h^2}\right)(1 + a^2u^2)^3 + 2Mu^3(1 + a^2u^2)^2 \tag{23}$$

With differentiating (23) respect to φ, easily can get

$$\left(\frac{d^2u}{d\varphi^2}\right) + u(1 + a^2u^2)^2 + 2a^2u^3 = \left(\frac{K^2 - 1}{h^2}\right)3a^2u(1 + a^2u^2)^2 + \left(\frac{6Ma^2u^2}{h^2}\right)(1 + a^2u^2)^2 + \frac{M}{h^2}(1 + a^2u^2)^3 + 3Mu^2(1 + a^2u^2)^2 + 4Ma^2u^4(1 + a^2u^2) \tag{24}$$

In approximation the parameters u^3 and greater orders are very small can vanish therefore

$$\left(\frac{d^2u}{d\varphi^2}\right) + u = 3a^2u\left(\frac{K^2 - 1}{h^2}\right) + \frac{M}{h^2} + 3Mu^2 + \left(\frac{6Ma^2u^2}{h^2}\right) \tag{25}$$

or in more familiar is as

$$\left(\frac{d^2u}{d\varphi^2}\right)+u\left(1-\frac{3a^2(K^2-1)}{h^2}\right)=\frac{M}{h^2}+3Mu^2\left(1+\frac{2a^2}{h^2}\right) \quad (26)$$

The result of Lagrangian method (26), is same with the equation of the Schwarzschild-like solution for ellipsoidal celestial objects (see [21]) which have been obtained by tensor analysis. The results in both tensor and Lagrangian methods for line element (8) are same.

AN APPLICATION AS ADVANCE OF PERIHELION

The theoretical analysis which derived in the previous section, is applying for an advance of perihelion of a secondary going around an ellipsoidal primary object.

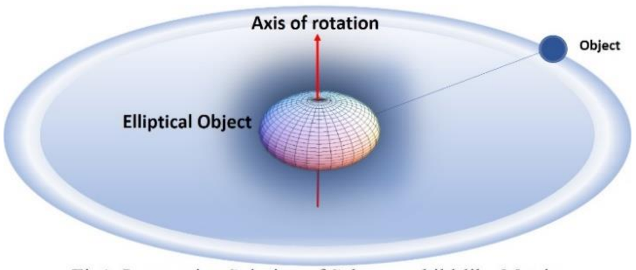

Fig1. Lagrangian Solution of Schwarzschild-like Metric

Here we start from equation (26) and rewrite as

$$\left(\frac{d^2u}{d\varphi^2}\right)+u=\frac{M}{h^2}+3Mu^2\left(1+\frac{2a^2}{h^2}\right)+\frac{3a^2}{h^2}(K^2-1)u \quad (27)$$

with

$$\frac{d\varphi}{d\tau}=\frac{h}{r^2+a^2} \quad \text{or} \quad h=(r^2+a^2)\frac{d\varphi}{d\tau} \quad (28)$$

For celestial objects such as stars and or planets, r is the radius and very large and $u=r^{-1}$ is small and consequently u^2 should be very small and therefore neglectable. As a first approximation, the small term $3mu^2\left(1+\frac{2a^2}{h^2}\right)$ can be neglected in the form of

$$\left(\frac{d^2u}{d\varphi^2}\right)+u=\frac{M}{h^2}+\frac{3a^2}{h^2}(K^2-1)u \quad (29)$$

The equation (29) is corresponding to the Newtonian theory and its solution is

$$u=\frac{M}{h^2}[1+e\cos(\varphi-\omega)] \quad (30)$$

Here e and ω are constant and define as eccentricity of the orbit and the initial longitude of the perihelion respectively. The parameter of e for values less than unit is the polar equation of an ellipse with the origin at one focus, and with Semi Latus Rectum $q=h^2/M$. Now for solution of (27) and to find more accuracy, we write,

$$\left(\frac{d^2u}{d\varphi^2}\right)+u-\frac{3a^2}{h^2}(K^2-1)u=\frac{M}{h^2}+3M\left(1+\frac{2a^2}{h^2}\right)u^2 \quad (31)$$

or

$$\left(\frac{d^2u}{d\varphi^2}\right)+\left[1-\frac{3a^2}{h^2}(K^2-1)\right]u=\frac{M}{h^2}+3M\left(1+\frac{2a^2}{h^2}\right)u^2 \quad (32)$$

If suppose

$$\beta=\frac{3a^2}{h^2}(K^2-1) \quad (33)$$

then we have

$$\left(\frac{d^2u}{d\varphi^2}\right)+(1-\beta)u=\frac{M}{h^2}+3M\left(1+\frac{a^2}{h^2}\right)u^2 \quad (34)$$

by choosing

$$1-\beta=K'^2 \quad \text{and} \quad \frac{M}{h^2}=\varepsilon K'^2 \quad (35)$$

then equation (34) will be as

$$\left(\frac{d^2u}{d\varphi^2}\right)+K'^2(u-\varepsilon)=3M\left(1+\frac{2a^2}{h^2}\right)u^2 \quad (36)$$

Now we try to solve the equation (36). For this, we suppose $3M\left(1+\frac{2a^2}{h^2}\right)u^2=0$ then, equation (36) is in the homogeny form

$$\left(\frac{d^2u}{d\varphi^2}\right)+K'^2(u-\varepsilon)=0 \quad (37)$$

The answer of this equation is

$$u=\frac{M}{h^2}[1+e\cos K'(\varphi-\omega)] \quad (38)$$

In attempting to solve and obtain the approximation the non-linear equation 26, the common approach is to substitute the linear solution of (37) into the right-hand side of equation (32) and then we get

$$\left(\frac{d^2u}{d\varphi^2}\right)+(1-\beta)u\frac{M}{h^2}+(3M)\left(1+\frac{2a^2}{h^2}\right)\left[\frac{M^2}{h^4}(1+e\cos K'(\varphi-\omega))\right]^2 \quad (39)$$

or

$$\left(\frac{d^2u}{d\varphi^2}\right)+(1-\beta)u=\frac{M}{h^2}+\left(\frac{3M^3}{h^4}\right)\left(1+\frac{2a^2}{h^2}\right)[1+e^2\cos^2 K'(\varphi-\omega)+2e\cos K'(\varphi-\omega)] \tag{40}$$

or

$$\left(\frac{d^2u}{d\varphi^2}\right)+(1-\beta)u=\frac{M}{h^2}+\left[\left(\frac{3M^3}{h^4}\right)+\left(\frac{3M^3e^2}{h^4}\right)\cos^2 K'(\varphi-\omega)+\left(\frac{6M^3e}{h^4}\right)\cos K'(\varphi-\omega)\right]\left(1+\frac{2a^2}{h^2}\right) \tag{41}$$

In the above equation out of the additional terms, the term which can produce any effect within the range of observations is the term containing $cosK'(\varphi-\omega)$ which here K' is the more additional factor so that have not studied yet. Since the particular integral of the equation

$$\left(\frac{d^2u}{d\varphi^2}\right)+u=A\cos\varphi \tag{42}$$

that

$$u_1=\frac{1}{2}A\,\varphi\sin\varphi \tag{43}$$

and the additional term

$$\left(\frac{6M^3e}{h^4}\right)\cos K'(\varphi-\omega)\left(1+\frac{2a^2}{h^2}\right) \tag{44}$$

gives a part of u given by

$$u_1=\left(1+\frac{2a^2}{h^2}\right)\left(\frac{3M^3}{h^4}\right)(eK'\varphi)\sin K'(\varphi-\omega) \tag{45}$$

Now the solution of equation (32) to the second order of approximation is, $u=u+u_1$ or,

$$u=\frac{M}{h^2}[1+e\cos K'(\varphi-\omega)]+\left[\left(\frac{3M^2}{h^2}\right)\left(1+\frac{2a^2}{h^2}\right)(eK'\varphi)\sin K'(\varphi-\omega)\right] \tag{46}$$

or

$$u=\frac{M}{h^2}+\left(\frac{Me}{h^2}\right)\left[\cos K'(\varphi-\omega)+\left(\frac{3M^2}{h^2}\right)\left(1+\frac{2a^2}{h^2}\right)(K'\varphi)\sin K'(\varphi-\omega)\right] \tag{47}$$

Substituting

$$\left(\frac{3M^2}{h^2}\right)\left(1+\frac{2a^2}{h^2}\right)(K'\varphi)=\delta'\omega \tag{48}$$

then we have

$$u=\frac{M}{h^2}+\left(\frac{Me}{h^2}\right)(\cos K'(\varphi-\omega)+\delta'\omega\sin K'(\varphi-\omega)) \tag{49}$$

Noting that $\delta'\omega$ is very small and positive term ($\delta'\omega>0$) so that $sin\delta'\omega=\delta'\omega=tan\delta'\omega$ and $cos\delta'\omega=1$, then we may write

$$u=\frac{M}{h^2}+\left(\frac{Me}{h^2}\right)[\cos K'(\varphi-\omega)+\tan(\delta'\omega)\sin K'(\varphi-\omega)] \tag{50}$$

or

$$u=\left(\frac{M}{h^2}\right)[1+e[\cos K'(\varphi-\omega)\cos\delta'\omega+\sin K'(\varphi-\omega)\sin(\delta'\omega)]] \tag{51}$$

or finally we have

$$u=\left(\frac{M}{h^2}\right)[1+e\cos(K'(\varphi-\omega)-\delta'\omega)] \tag{52}$$

for $\varphi=2\pi$ then $\delta'\omega$ is in the form of

$$\delta'\omega=\left(\frac{6\pi M^2}{h^2}\right)\left(1+\frac{2a^2}{h^2}\right)K' \tag{53}$$

using the standard relation

$$\frac{M}{h^2}=\frac{1}{l}=\frac{1}{b(1-e^2)} \tag{54}$$

where here b is the mean radial distance of a secondary. From Kepler's second law the period T is

$$T=\frac{2\pi}{\sqrt{M}}b^{3/2} \tag{55}$$

Which this period is orbital period of the secondary from which

$$M=\frac{4\pi^2b^3}{T^2} \tag{56}$$

Therefore, from equation (53) we get

$$\delta'\omega=\left(\frac{6\pi M^2}{h^2}\right)\left(1+\frac{2a^2}{h^2}\right)K' \tag{57}$$

so that

$$h^2=Mb(1-e^2) \tag{58}$$

$$\delta'\omega=\left[\frac{6\pi M}{b(1-e^2)}\right]\left(1+\frac{a^2}{mb(1-e^2)}\right) \tag{59}$$

then for equation (49) we obtain Final and corrected

$$\delta'\omega = \frac{24\pi^3 b^2}{T^2(1-e^2)}\left[1 + \frac{a^2T^2}{2\pi^2 b^4(1-e^2)}\right]^{\frac{1}{2}} \tag{60}$$

This is a new relation in terms of a , b ,e and period T of a mass orbiting around an elliptical shape as primary and very massive object. The bracket in the right side of equation (60), is a new and small term which will produce by shape of primary object in the center. For $(a = 0)$ the value of $\delta'\omega$ is as Schwarzschild perihelion or in Newtonian form. The shape of object in this case is therefore in the spherical equation (51) be in the following form.

$$\delta'\omega = \frac{24\pi^3 b^2}{T^2(1-e^2)} \tag{61}$$

THE BENDING OF LIGHT

Here to consider the bending of light around gravitational field of an elliptical object we start from equation (15). In approximation u^3 and its greater orders are very small and therefore can be vanish. After rearranging equation (15) we have,

$$\left(\frac{d^2u}{d\varphi^2}\right) + u = \frac{M}{h^2} + 3Mu^2\left(1 + \frac{2a^2}{h^2}\right) + \frac{3a^2}{h^2}u(K^2-1) \tag{62}$$

For the track of a light ray with $ds = 0$ and therefore $h = \infty$ Hence the track of light ray for the neighbourhood of a gravitating ellipsoid mass m is,

$$\frac{d^2u}{d\varphi^2} + u = 3Mu^2 \tag{63}$$

by neglecting the small term $3Mu^2$ to the first approximation, we have

$$\frac{d^2u}{d\varphi^2} + u = 0 \tag{64}$$

The general solution of this equation is in the form of $u(\varphi) = Acos\varphi + Bsin\varphi$. Here A and B are constants therefore $du/d\varphi = -Acos\varphi + Bsin\varphi$ with boundary conditions $\varphi = 0$, $u = 1/r$ and $du/d\varphi = 0$ we get $A = 1/r$ and $B = 0$. With substituting these values in the $u(\varphi)$ have,

$$u = \left(\frac{1}{r}\right)\cos\varphi \tag{65}$$

For the second approximation, inserting u from Eq(63) and then we get

$$\frac{d^2u}{d\varphi^2} + u = \left(\frac{3M}{r^2}\right)\cos^2\varphi \tag{66}$$

The particular solution of the above equation is,

$$u_1 = \left(\frac{1}{1+D^2}\right)\left(\frac{3M}{r^2}\cos^2\varphi\right) \tag{67}$$

where $D^2 = (d/d\varphi)$ and finally,

$$u_1 = \frac{M}{r^2}(\cos^2\varphi + 2\sin^2\varphi) \tag{68}$$

Hence the complete solution of equation (63) to the second approximation is

$$u_1 = \frac{1}{r}cos\varphi + \frac{M}{r^2}(\cos^2\varphi + 2\sin^2\varphi) \tag{69}$$

multiplying both side of the equation (69) by (R/u),we get,

$$u_1 = \left(\frac{cos\varphi}{u}\right) + \frac{M}{r}\left(\frac{\cos^2\varphi + 2\sin^2\varphi}{u}\right) \tag{70}$$

By substituting $r = 1/u$ we have,

$$R = (rcos\varphi) + \frac{M}{R}(r\cos^2\varphi + 2r\sin^2\varphi) \tag{71}$$

with choosing $\theta = \pi/2$ in x , y and z then,

$$x = \sqrt{r^2 + a^2}cos\varphi \tag{72}$$

$$y = \sqrt{r^2 + a^2}sin\varphi \tag{73}$$

and after substituting these values in in equation (70) we have,

$$R = \left(\frac{x\sqrt{x^2+y^2-a^2}}{\sqrt{x^2+y^2}}\right) + \left(\frac{M}{R}\right)\left[\frac{(x^2+2y^2)}{x^2+y^2}\sqrt{x^2+y^2-a^2}\right] \tag{74}$$

or

$$R = \left(\frac{\sqrt{x^2+y^2-a^2}}{\sqrt{x^2+y^2}}\right) + \left[x + \left(\frac{M}{R}\right)\frac{(x^2+2y^2)}{\sqrt{x^2+y^2}}\right] \tag{75}$$

The second term in the bracket in equation (75) measures very slight deviation from the straight-line path $x = R$. For y very large as compared to x the above equation can be as

$$R = \left[\sqrt{1 - \frac{a^2}{x^2+y^2}}\right] + \left[x + \frac{M}{R}(\pm 2y)\right] \tag{76}$$

or

$$R = \beta\left[x + \frac{M}{R}(\pm 2y)\right] \tag{75}$$

where

$$\beta = \sqrt{1 - \frac{a^2}{(x^2 + y^2)}} \tag{76}$$

Here a is small and a^2 smaller and consequently $(x^2 + y^2)$ large. Therefore $a^2/(x^2 + y^2)$ is small and always less than 1. Hence the factor β is a small but it is not a negligible term for an elliptical shape. For $\beta = 1$, equation (76) can express in spherical form of star and consequently for $\beta < 1$, it is explaining an ellipsoid form of a star and or any celestial object.

CONCLUSION

Elliptical galaxies represent approximately 10% of observed galaxies.

The bending of light rays near massive objects, already have calculated by Schwarzschild line element, so that it is valid only for spherical objects. For non-spherical or preferably elliptical shape, such as galaxies, stars or planets required some modification in the line element and geometry of space-time. For this reason, the line element [4] presented for elliptical object. The solution of this line element has done with Lagrangian method as well as tensor method. We also calculated the bending of light rays near such these objects (elliptical object). The important consequence of this calculation as a test of general theory of relativity for bending of light rays around elliptical objects, show a small additional term which describe for non-spherical or elliptical objects. However, this additional term is very small but it is not negligible term and for very massive elliptical objects it is important. Moreover, for $a = 0$ the Schwarzschild line element for spherical objects will be produce.

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