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λ-Symmetry and Integrating Factor For $\ddot{x}(f(t,x) + g(t,x)\dot{x})e^x$

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ABSTRACT

in this paper, we will calculate an integrating factor, first integral and reduce the order the non-Linear second-order ODEs $\ddot{x}(f(t,x) + g(t,x)\dot{x})e^x$, through λ-symmetry method. Moreover, we compute an integrating factor, first integral and reduce the order for particular cases of this equation.

Keyword: Symmetry, Integrating Factor, First Integral, Order reduction

INTRODUCTION

Symmetries method have been widely used to reduce the order of an ordinary differential equation (ODE) and to reduce the number of independent variables in a partial differential equation (PDE)[1].
There are many examples of ODEs that have trivial Lie symmetries. In 2001, Muriel and Romero introduced λ-symmetry method to reduce the order of an ODEs and to find general solutions for such examples.
Recently, they [2] presented techniques to obtain first integral, integrating factor, λ-symmetry of second-order ODEs $\ddot{x} = F(t,x,\dot{x})$ and the relationship between them.
In addition, the study of a λ-symmetry method of the ODEs permits us the de termination of an integrating factor and reduce the order of the ODEs and explain the reduction process of many ODEs that lack Lie symmetries.
In this paper, first we will recall some of the foundational results about symmetry and λ-symmetry rather briefly. we present some theorems about an integrating factor, first integral and reduce the order of the ODEs. second, we will calculate an integrating factor, first integral and reduce the or der the non-Linear second-order ODEs $\ddot{x} = (f(t,x) + g(t,x)\dot{x})e^x$, through λ symmetry method, which are non-Lie symmetry equation and functions $f(t,x)$ and $g(t,x)$ are arbitrary.
Moreover, we will reduce the order particular cases of the equation $x\ddot{} = (f(t,x) + g(t,x)\dot{x})e^x$, which are $\ddot{x} = (f(t,x) + g(t,x)\dot{x})e^x$, and $\ddot{x} = (f(t,x) + g(t,x)\dot{x})e^x$, through λ-symmetry method. we will present many examples for these equations.

λ-SYMMETRIES ON ODES

In this section we recall some of the foundational results about symmetry and λ-symmetry rather briefly [2-9].
Let $\mathbf{v}$ be a vector field defined on an open subset $M \subset T \times X$.

We denote by $M^{(n)}$ the corresponding jet space $M^{(n)} \subset T \times X^{(n)}$, for $n \in N$. Their elements are $(t, x^{(n)}) = (t, x, x_1, \cdots, x_n)$, where, for $i = 1, 2, \cdots, n, x_i$ denotes the derivative of order i of x with respect to t. Suppose

$$\Delta(t,\ x^{(n)}) = 0 \tag{1}$$

be an ODE defined over the total space M. The latter characterizes a Lie symmetry of an ODE as a vector filed $\mathbf{v}= \xi(t,x)\partial/\partial t + \eta(t,x)\partial/\partial x$, that satisfies $\mathbf{v}^{(n)}[\Delta(t,x^{(n)})] = 0$, if $\Delta(t,x^{(n)}) = 0$, where $\mathbf{v}^{(n)}$that called $n-th$ prolongation of $\mathbf{v}$ is

$$\mathbf{v}^{(n)} = \xi(t,x)\frac{\partial}{\partial t} + \eta(t,x)\frac{\partial}{\partial x} + \sum_{i=1}^{n} \eta^{(i)}\left(t, x^{(i)}\right)\frac{\partial}{\partial x_i}$$

Where

$$\eta^{(i)}\left(t,x^{(i)}\right) = D_t\left(\eta^{(i-1)}\left(t,x^{(i-1)}\right)\right) - D_t(\xi(t,x))x_i$$

and $\eta^{(0)}(t,x) = \eta(t,x)$ for $i = 1, \cdots, n$, where D_t denote the total derivative operator with respect to t [9].

If an ODE does not have Lie point symmetry, then we using λ-symmetry method for reduce of order the ODE. λ-symmetry method is as follows [3].

For every function $\lambda \in C^{\infty}(M^{(1)})$, we will define a new prolongation and Lie symmetry of v in the following way.

Let $\mathbf{v}= \xi(t,x)\partial/\partial t + \eta(t,x)\partial/\partial x$, be a vector field defined on M, and let $\lambda \in C^{\infty}(M^{(1)})$ be an arbitrary function. The λ-prolongation of order n of $\mathbf{v}$, denoted by $\mathbf{v}^{[\lambda,(n)]}$, is the vector field defined on M by

$$\mathbf{v}^{[\lambda,(n)]} = \xi(t,x)\frac{\partial}{\partial t} + \eta(t,x)\frac{\partial}{\partial x} + \sum_{i=1}^{n} \eta^{(i)}\left(t, x^{(i)}\right)\frac{\partial}{\partial x_i}$$

where

$$\eta^{[\lambda,(i)]}\left(t,x^{(i)}\right) = (D_t + \lambda)\left(\eta^{[\lambda,(i-1)]}\left(t,x^{(i-1)}\right)\right) - \left((D_t+\lambda)\xi(t,x)\right)x_i$$

and $\eta^{[\lambda,(0)]}(t,x) = \eta(t,x)$ for $i = 1, \cdots, n$.

A vector field $\mathbf{v}$ is a λ-symmetry of the Eq. (1), if there exists function $\lambda \in C^{\infty}(M^{(1)})$, such that $\mathbf{v}^{[\lambda,(n)]}[\Delta(t,x^{(n)})] = 0$, if $\Delta(t,x^{(n)}) = 0$.

Note. Suppose vector field $v = \partial/\partial x$ be a λ-symmetry of the Eq.(1), then

$$\eta^{[\lambda,(n-1)]} = \frac{\partial}{\partial x} + (D_t + \lambda)(1)\frac{\partial}{\partial x_1} + (D_t+\lambda)(D_t+\lambda)(1)\frac{\partial}{\partial x_2} + \cdots + (D_t+\lambda)(D_t+\lambda)(1)\ \frac{\partial}{\partial x_{n-1}}$$

or equivalent

$$v^{[\lambda,(n-1)]} = \sum_{i=1}^{n} (D_t+\lambda)^{(i)}(1)\frac{\partial}{\partial x_i} \tag{2}$$

An integrating factor of the Eq. (1), is a function $\mu(t,x^{(n-1)})$ such that the equation $\mu.\Delta = 0$ is an exact equation,

$$\mu\left(t,x^{(n-1)}\right).\Delta\left(t,x^{(n)}\right) = D_t\left(G\left(t,x^{(n-1)}\right)\right).$$

Function $G\left(t,x^{(n-1)}\right)$, will be called a first integral of the Eq. (1), and $D_t\left(G\left(t,x^{(n-1)}\right)\right) = 0$, is a conserved form of the Eq.(1) [6, 10]. Let

$$x_n = F(t,x^{(n-1)}) \tag{3}$$

be a nth-order ordinary differential equation, where F is an analytic function of its arguments. We denote by $A = \partial_t + x_1\partial_x + x_2\partial_{x^{(1)}} + \cdots + F(t,x^{(n-1)})\partial_{x^{(n-1)}}$ the vector field associated with (3) [3].

Function $I(t,x^{(n-1)})$ is a first integral [7]of (3), such that $A(I) = 0$ and an integrating factor of (3), is any function $\mu(t,x^{(n-1)})$ such that

$$\mu\left((t,x^{(n-1)})x^{(n)} - F\left(t,x^{(n-1)}\right)\right) = D_t I(t,x^{(n-1)}).$$

By using (2), It can be checked that the vector field $v = \partial_x$ is a λ-symmetry of (3), if the function $\lambda(t,\ x^{(k)})$ is any particular solution of the equation

$$(D_t+\lambda)^{(n)}(1) = \sum_{i=0}^{n-1} (D_t+\lambda)^{(i)}(1)\frac{\partial F}{\partial x_i} \tag{4}$$

Theorem 2.1. *If $I(t,\ x^{(n-1)})$ is a first integral of (3), then $\mu\left(t,x^{(n-1)}\right) = I_{x^{(n-1)}}\ (t,x^{(n-1)})$is an integrating factor of (3).*

Proof. Let $I(t,\ x^{(n-1)})$ be a first integral of (3), then

$$A(I) = I_t + x^{(1)}I_x + x^{(2)}I_{x^{(1)}} + \cdots + F\left(t,x^{(n-1)}\right)I_{x^{(n-1)}} = 0.$$

Therefore

$$I_t + x^{(1)} I_x + x^{(2)} I_{x^{(1)}} + \cdots + x^{(n-1)} I_{x^{(n-2)}} = -F\big(t, x^{(n-1)}\big) I_{x^{(n-1)}}$$

and

$$D_t I = I_t + x^{(1)} I_x + x^{(2)} I_{x^{(1)}} + \cdots + x^{(n-1)} I_{x^{(n-2)}} + x^{(n)} I_{x^{(n-1)}} = -F\big(t, x^{(n-1)}\big) I_{x^{(n-1)}} + x(n) I_{x^{(n-1)}} = I_{x^{(n-1)}} \big(x^{(n)} - F(t, x^{(n-1)})\big).$$

Hence $\mu(t, x^{(n-1)}) = I_{x^{(n-1)}}(t, x^{(n-1)})$. The vector field $v = \xi(t,x)\partial_t + \eta(t,x)\partial_x$ is a λ-symmetry of equation (3) if and only if $[v^{[\lambda,(n-1)]}, A] = \lambda v^{[\lambda,(n-1)]} + \tau A$ where $\tau = -(A + \lambda)(\xi(t,x))$ [3]. When $v = \partial_x$ is a λ-symmetry of equation 3) if and only if $\left[v^{[\lambda,(n-1)]}, A\right] = \lambda v^{[\lambda,(n-1)]}$.

Theorem 2.2. *If $v = \partial_x$ is a λ-symmetry of (3) for some function $\lambda(t, x^{(n-1)})$, then there is a first integral $I(t, x^{(n-1)})$ of (3) such that $v^{[\lambda,(n-1)]}(I) = 0$*

Proof. If $v = \partial_x$ is a λ-symmetry of (3) for some function $\lambda(t, x^{(n-1)})$, then $[v^{[\lambda,(n-1)]}, A] = \lambda v^{[\lambda,(n-1)]}$.
Therefore $\{v^{[\lambda,(n-1)]}, A\}$ is an involutive set of vector fields in $M^{(n-1)}$ and there is function $I(t, x^{(n-1)})$ such that $v^{[\lambda,(n-1)]}(I) = 0$ and $A(I) = 0$.
Let $\omega(t, x^{(n-1)})$ be a first integral of $v^{[\lambda,(n-1)]}$, i.e, $v^{[\lambda,(n-1)](\omega)} = 0$, then by using of (2), $\omega(t, x^{(n-1)})$ is a solution of PDE:

$$\omega_x + (D_t + \lambda)(1)\omega_{x^{(1)}} + \cdots + (D_t + \lambda)^{n-1}(1)\omega_{x^{(n-1)}} = 0 \tag{5}$$

Let $I\big(t, x^{(n-1)}\big) = G(t, \omega(t, x^{(n-1)}))$ be a first integral of (3), then

$$\begin{aligned} 0 = A(I) &= I_t + x^{(1)} I_x + x^{(2)} I_{x^{(1)}} + \cdots + F\big(t, x^{(n-1)}\big) I_{x^{(n-1)}} \\ &= (G_t + G_\omega \omega_t) + x^{(1)}(G_\omega \omega_x) + x^{(2)}\big(G_\omega \omega_{x^{(1)}}\big) + \cdots + F(t, x^{(n-1)})(G_\omega \omega_{x^{(n-1)}}) \\ &= G_t + \omega t + x(1)\omega x + x(2)\omega x(1) + \cdots + F(t, x^{(n-1)})\omega x(n-1))G\omega \\ &= Gt + A(\omega)G\omega = G_t + H(t,\omega)G_\omega \end{aligned}$$

where $A(\omega) = H(t,\omega)$. Hence, if $G(t,\omega)$ is a particular solution of $Gt + H(t,\omega)G_\omega = 0$ then $I\big(t, x^{(n-1)}\big) = G(t, \omega(t, x^{(n-1)}))$ is a first integral of (3). In summary, a procedure to find a first integral $I(t, x^{(n-1)})$ and consequently an integrating factor $\mu(t, x^{(n-1)})$ of (3), by using λ-symmetry method is as follows.

- The vector field $v = \partial_x$ is a λ-symmetry of (3), if function $\lambda(t, x^{(n-1)})$ is any particular solution of the equation (4).
- Find a first integral $\omega(t, x^{(n-1)})$, i.e. a particular solution of the equation (5).
- Evaluate $A(\omega) = H(t, \omega)$.
- Find a first integral $G(t,\omega)$ from the solution of the equation $G_t + H(t, \omega)G_\omega = 0$.
- The function $I\big(t, x^{(n-1)}\big) = G(t, \omega(t, x^{(n-1)}))$ is a first integral of (3).
- The function $\mu\big(t, x^{(n-1)}\big) = I_{x^{(n-1)}}(t, x^{(n-1)})$ is an integrating factor of (3).

We focus our attention on second order ODEs, $n = 2$ in equation (3), i.e.

$$\ddot{u} = F(x, u, \dot{u}) \tag{6}$$

where F is an analytic function of its arguments. A procedure to find a first integral $I(t, x, \dot{x})$ and consequently an integrating factor $\mu(t, x, \dot{x})$ of (3), by using λ-symmetry method is as follows.

- The vector field $v = \partial_x$ is a λ- symmetry of (6), if function $\lambda(t, x, \dot{x})$ is any particular solution of the equation

$$D_t(\lambda) + \lambda^2 = \frac{\partial F}{\partial x} + \lambda \frac{\partial F}{\partial x} \tag{7}$$

- Let v be a λ-symmetry of (6), then $\omega(t, x, \dot{x})$ is a first-order invariant of $v^{[\lambda,1]}$, that is, any particular solution of the equation

$$\omega_x + \lambda(t, x, \dot{x}).\omega\dot{x} = 0 \tag{8}$$

- Evaluate $A(\omega) = H(t, \omega)$.
- Find a first integral $G(t, \omega)$ from the solution of the equation $G_t + H(t, \omega)G_\omega = 0$.
- The function $I(t, x, \dot{x}) = G(t, \omega(t, x, \dot{x}))$ is a first integral of (6).
- The function $\mu(t, x, \dot{x}) = Ix^{\cdot}(t, x, \dot{x})$ is an integrating factor of (6).

REDUCTION OF $\ddot{x} = (f(t,x) + g(t,x)\dot{x})e^x$, BY λ-SYMMETRY METHOD

Let

$$x^{\cdot\cdot} = (f(t,x) + g(t,x)\dot{x})e^x \tag{9}$$

be a second-order ordinary differential equation, where $F(t,x,\dot{x}) = (f(t,x) + g(t,x)\dot{x})e^x$ is an analytic function on its arguments and $f(t,x)$ and $g(t,x)$ are arbitrary functions. It can be checked that this equation does not have Lie point symmetry. There exists a function $\lambda(t,x,\dot{x})$ such that the vector field $v = \partial_x$ is a λ-symmetry of the equation (9). To determine such functions $\lambda(t,x,\dot{x})$, by (7), λ is any particular solution for the equation.

$$0 = D_t(\lambda) + \lambda^2 - \frac{\partial F}{\partial x} - \lambda \frac{\partial F}{\partial \dot{x}}$$
$$= \lambda_t + \dot{x}\lambda_x + \ddot{x}\lambda_{\dot{x}} + \lambda^2 - (f_x + g_x\dot{x})e^x - (f + g\dot{x})e^x - \lambda g e^x$$

or corresponding to

$$\lambda_t + \dot{x}\lambda_x + (f + g\dot{x})e^x\lambda_{\dot{x}} + \lambda^2 - (f_x + g_x\dot{x})e^x - (f + g\dot{x})e^x - \lambda g e^x = 0 \tag{10}$$

For the sake of simplicity, we try to find a solution λ (10) of the form $\lambda(t,x,\dot{x}) = \lambda_1(t,x)\dot{x} + \lambda_2(t,x)$, we obtain the following system:

$$\lambda_1^2 + (\lambda_1)_x = 0$$
$$(\lambda_1)_t + (\lambda_2)_x + 2\lambda_1\lambda_2 - g_x e^x - g e^x = 0,$$
$$(\lambda_2)_t + f e^x \lambda_1 + \lambda_2^2 - f_x e^x - f e^x - \lambda_2 g e^x = 0,$$

A particular solution of the first equation is given by $\lambda_1 = 0$. The second and third equations become

$$(\lambda_2)_x - g_x e^x - g e^x = 0$$
$$(\lambda_2)_t + \lambda_2^2 - f_x e^x - f e^x - \lambda_2 g e^x = 0$$

For the first equation and the second equation, we have

$g = (\lambda_2 + 1)e^{-x}$, and $f = \left(\int ((\lambda_2)_t - \lambda_2)dx\right)e^{-x}$

A particular solution of this system is $\lambda_2 = ge^x - 1$, where $g_t - g + e^{-x} = f_x + f$. Hence,

$$\lambda(t,x,\dot{x}) = \lambda_1(t,x)\dot{x} + \lambda_2(t,x) = \lambda_2(t,x) = g(t,x)e^x - 1.$$

Therefore, the vector field $v = \partial_x$ is a λ-symmetry of (9) for

$$\lambda(t,x,\dot{x}) = g(t,x)e^{\,x} - 1 \tag{11}$$

To find an integrating factor associated to λ, first, we find a first integral invariant $\omega(t,x,\dot{x})$ of $v^{[\lambda,1]}$ by the equation that corresponds to (8), which means,

$$\omega_x + (ge^x - 1)\omega\dot{x} = 0 \tag{12}$$

For the sake of simplicity, we try to find a solution ω of the form $\omega(t,x,\dot{x}) = \omega_1(t,x)\dot{x} + \omega_2(t,x)$, we have

$$(\omega_1)_x\dot{x} + (\omega_2)_x + (ge^x - 1)\omega_1 = 0$$

or corresponding

$$(\omega_1)_x = 0, \qquad (\omega_2)_x + (ge^x - 1)\omega_1 = 0$$

A particular solution of the first equation is given by $\omega_1 = 1$. the second equation become $(\omega_2)_x + (ge^x - 1) = 0$, the solution of this equation is $\omega_2 = -\int ge^x dx + x$. Hence,

$$\omega(t,x,\dot{x}) = \omega_1(t,x)\dot{x} + \omega_2(t,x)\dot{x} - \int g(t,x)e^x dx + x \tag{13}$$

is a particular solution for (12). The vector field associated $A = \partial_t + \dot{x}\partial_x + F(t,x,\dot{x})\partial\dot{x}$ acts on ω, then, we have

$$A(\omega) = -\int e^x g_t dx + \dot{x} + fe^x$$
$$= -\int e^x g_t dx + \dot{x} + \left(\int ((\lambda_2)_t - \lambda_2)\,dx\right)$$
$$= -\int e^x g_t dx + \dot{x} + \int (g_t e^x - ge^x + 1)\,dx$$
$$= \dot{x} - \int ge^x dx + x = \omega = H(t,\omega)$$

Therefore, $A(\omega) = \omega = H(t,\omega)$. The function

$$G(t,\omega) = \omega e^{-t} \tag{14}$$

is a particular solution for the equation $G_t + \omega G_\omega = 0$. Therefore,

$$I(t,x,\dot{x}) = G\big(t,\omega(t,x,\dot{x})\big) = (\,\dot{x} + x - \textstyle\int g(t,x)e^x dx)e^{-t}$$

is a first integral of (9) also the function

$$\mu(t,x,\dot{x}) = I_{\dot{x}}(t,x,\dot{x}) = e^{-t} \tag{15}$$

is an integrating factor of (9). Also,

$$\begin{aligned} D_t\big(G(t,\omega(t,x,\dot{x}))\big) &= D_t\big(\dot{x} + x - \textstyle\int g(t,x)e^x dx\big)e^{-t}\big) \\ &= 0 \end{aligned}$$

is a conserved form of (9).

Summation. λ-symmetry method to find a first integral $I(t,x,\dot{x})$ and con sequently an integrating factor $\mu(t,x,\dot{x})$ of (9) is as follows.

• The vector field $v = \partial_x$ is a λ- symmetry of (9), and function $\lambda(t,x,\dot{x}) = g(t,x)e^x - 1$ is a particular solution of the equation $D_t(\lambda) + \lambda^2 = \partial F/\partial x + \lambda\, \partial F/\partial \dot{x}$.
• Let v be a λ-symmetry of (9), then $\omega(t,x,\dot{x}) = \dot{x} + x - \int g(t,x)e^x dx$ is a first-order invariant of $v^{[\lambda,1]}$, that is, a particular solution of the equation $\omega_x + (g(t,x)e^x - 1)\omega x^{\cdot} = 0$.
• We have $A(\omega) = H(t,\omega) = \omega$.
• The function $G(t,\omega) = \omega e^{-t}$ is a particular solution for the equation $G_t + \omega G_\omega = 0$.
• The function $I(t,x,\dot{x}) = G\big(t,\omega(t,x,\dot{x})\big) = (\dot{x} + x - \int g(t,x)e^x dx)e^{-t}$ is a first integral of (9). Also, $D_t(G\big(t,\omega(t,x,\dot{x})\big) = D_t\left((\dot{x} + x - \int g(t,x)e^x dx)e^{-t}\right) = 0$, is a conserved form of (9).
• The function $\mu(t,x,\dot{x}) = I\dot{x}(t,x,\dot{x}) = e^{-t}$, is an integrating factor of (9).

Corollary 3.1. *Equality* $D_t\left((\dot{x} + x - \int g(t,x)e^x dx)e^{-t}\right) = 0$ *is a conserved form of (9), therefore reduce the order of equation* $\ddot{x} = (f(t,x) + g(t,x)\dot{x})e^x$ *is the equation* $\dot{x} + x - \int g(t,x)e^x dx = 0$.

SPECIAL CASES OF THE EQUATION $\ddot{x} = (f(t,x) + g(t,x)\dot{x})e^x$

Special cases of the equation $\ddot{x} = (f(t,x) + g(t,x)\dot{x})e^x$ are $\ddot{x} = (f(x) + g(x)\dot{x})e^x$. We consider the second-order ODE

$$\ddot{x} = (f(t) + g(t)\dot{x})e^x \tag{16}$$

where $F(t,x,\dot{x}) = (f(t) + g(t)\dot{x})e^x$ is an analytic function on its arguments and $f(t)$ and $g(t)$ are arbitrary functions. It can be checked that this equation does not have Lie point symmetry.
Similar of the equation (9), λ-symmetry method to find a first integral $I(t,x,\dot{x})$ and consequently an integrating factor $\mu(t,x,\dot{x})$ of (16) is as follows: The vector field $v = \partial_x$ is a λ-symmetry of (16), and function $\lambda(t,x,x^{\cdot}) = 1/t + g(t)e^x$ is a particular solution of the equation $D_t(\lambda) + \lambda^2 = \partial F/\partial x + \lambda\, \partial F/\partial x^{\cdot}$.
Let v be a λ-symmetry of (16), then $\omega(t,x,\dot{x}) = \dot{x} - g(t)e^x - x/t$ is a first-order invariant of $v^{[\lambda,1]}$, that is, a particular solution of the equation $\omega_x + (1/t + g(t)e^x)\omega\dot{x} = 0$. We have $A(\omega) = H(t,\omega) = -(1/t)\omega$.
The function $G(t,\omega) = t\omega$ is a particular solution for the equation $G_t - (1/t)\,\omega\, G_\omega = 0$. The function $I(t,x,\dot{x}) = G\big(t,\omega(t,x,\dot{x})\big) = t\dot{x} - tg(t)e^x - x$, is a first integral of (16).
Also, $D_t(G\big(t,\omega(t,x,\dot{x})\big) = D_t(t\dot{x} - tg(t)e^x - x) = 0$, is a conserved form of (16). The function $\mu(t,x,\dot{x}) = I_{\dot{x}}(t,x,\dot{x}) = t$ is an integrating factor of (16).

Corollary 4.1. *Equality* $D_t(t\dot{x} - tg(t)e^x - x) = 0$, *is a conserved form of (4.1), therefore reduce the order of the equation* $\ddot{x} = (f(t) + g(t)\dot{x})e^x$, *is the equation* $t\dot{x} - tg(t)ex - x = 0$.

We consider the second-order ODE

$$\ddot{x} = (f(t) + g(t)\dot{x})e^x \tag{17}$$

where $F(t,x,\dot{x}) = (f(x) + g(x)\,\dot{x})e^x$ is an analytic function on its arguments and $f(x)$ and $g(x)$ are arbitrary functions. It can be checked that this equation does not have Lie point symmetry.
Similar of the equation (9), λ-symmetry method to find a first integral $I(t,x,\dot{x})$ and consequently an integrating factor $\mu(t,x,\dot{x})$ of (17) is as follows: The vector field $v = \partial_x$ is a λ-symmetry of (17), and function $\lambda(t,x,\dot{x}) = g(x)e^x - 1$ is a particular solution of the equation $D_t(\lambda) + \lambda^2 = \partial F/\partial x + \lambda \partial F/\partial \dot{x}$,
Let v be a λ-symmetry of (17), then $\omega(t,x,\dot{x}) = \dot{x} + x - \int g(x)e^x dx$ is a first-order invariant of $v^{[\lambda,1]}$, that is, a particular solution of the equation $\omega_x + (g(x)e^x - 1)\omega_{\dot{x}} = 0$. We have $A(\omega) = H(t,\omega) = \omega$.
The function $G(t,\omega) = \omega e^{-t}$, is a particular solution for the equation $G_t + \omega G_\omega = 0$. The function $I(t,x,\dot{x}) = G\big(t,\omega(t,x,\dot{x})\big) = (\dot{x} + x - \int g(x)e^x dx)e^{-t}$, is a first integral of (4.2). Also, $D_t(G\big(t,\omega(t,x,\dot{x})\big) = D_t\left((\dot{x} + x - \int g(x)e^x dx)e^{-t}\right) = 0$, is a conserved form of (17). The function $\mu(t,x,\dot{x}) = I\,\dot{x}(t,x,\ \dot{x}) = e^{-t}$, is an integrating factor of (17).

Corollary 4.2. *Equality* $D_t(\dot{x} + x - \int g(x)e^x dx) = 0$, *is a conserved form of (4.2), therefore reduce the order of the equation* $\ddot{x} = (f(x) + g(x)\dot{x})e^x$, *is the equation* $\dot{x} + x - \int g(x)e^x dx = 0$.

SOME ILLUSTRATIONS

Example 1. We consider the second-order ordinary differential equation

$$\ddot{x} = (t^3 - 1)cos\, x + (tsin\, x + 1)\dot{x} \qquad (18)$$

where in the Eq. (9), $f(t,x) = (t^3 - 1)\, cos\, xe^{-x}$ and $g(t,x) = (t\, sin\, x + 1)e^{-x}$ and also the function $F(t,x,\dot{x}) = (t^3 - 1)cos\, x + (tsin\, x + 1)\dot{x}$, is an analytic function on its arguments. It can be checked that this equation does not have Lie point symmetry. Therefore, we have for the equation (18).
The vector field $v = \partial_x$ is a λ-symmetry of (18), and function $\lambda(t,x,\dot{x}) = g(t,x)e^x - 1 = tsin\, x$, is a particular solution of the equation $D_t(\lambda) + \lambda^2 = \partial F/\partial x\ +\ \lambda \partial F/\partial\, \dot{x}$.
Let v be a λ-symmetry of (5.1), then $\omega(t,x,\ \dot{x}) = \ \dot{x} + x - \int g(t,x)e^x dx = \ \dot{x} + t\, cos\, x$, is a first-order invariant of $v^{[\lambda,1]}$, that is, a particular solution of the equation ω_x+($t$sin x)$\omega_{\dot{x}}$ = 0. We have $A(\omega) = H(t,\omega) = \omega$. The function $G(t,\ \omega) = \omega e^{-t}$, is a particular solution for the equation $G_t + \omega G_\omega = 0$.
The function

$$I(t,x,\dot{x}) = G\big(t,\omega(t,x,\dot{x})\big) =$$

$$(\ \dot{x} + x - \int g(t,x)e^x dx)e - t\ = (\ \dot{x} + t\, cos\, x)e^{-t},$$

is a first integral of (5.1). $D_t(G(t,\omega(t,x,\dot{x})) = Dt((\dot{x} + t\, cos\, x)e^{-t}) = 0$, is a conserved form of (18). The function $\mu(t,x,\dot{x}) = I_{\dot{x}}(t,x,\dot{x}) = e^{-t}$, is an integrating factor of (18). Therefore, we reduce the order of the equation $\ddot{x} = (t^3 - 1)cos\, x + (tsin\, x + 1)\dot{x}$, to the equation $(\dot{x} + t\, cos\, x)e^{-t} = 0$. This equation does not have Lie point symmetries.

Example 2. Let

$$\ddot{x} = (sinh\, t + cosh\, t/t + cosh\, t\dot{x})e^x \qquad (19)$$

where in the Eq. (19), $f(t) = sin\, t + cosh\, t/t$ and $g(t) = cosh\, t$ and also the function $F(t,x,\dot{x}) = (sin\, t + cosht/t + cosh\, t\dot{x})e^x$, is an analytic function on its arguments.
This equation does not have Lie point symmetry. We have for the equation (19).
The vector field $v = \partial_x$ is a λ-symmetry of (19), and function $\lambda(t,x,\dot{x}) = 1/t + g(t)e^x = 1/t + \, cosh(t)e^x$, is a particular solution of the equation $D_t(\lambda) + \lambda^2 = \partial F/\partial x\ +\ \lambda\partial F/\partial \dot{x}$.
Let v be a λ-symmetry of (19), then $\omega(t,x,\dot{x}) = \dot{x} - g(t)e^x - x/t = \dot{x} - cosh\, te^x - x/t$, is a first-order invariant of $v^{[\lambda,1]}$, that is, a particular solution of the equation $\omega_x + (1/t + cosh\, te^x)\omega x\dot{} = 0$. We have $A(\omega) = H(t,\omega) = -(1/t)\omega$.
The function $G(t,\omega) = t\omega$, is a particular solution for the equation $G_t - (1/t)\omega G_\omega = 0$. The function $I(t,x,\dot{x}) = G\big(t,\omega(t,x,\dot{x})\big) = t\dot{x} - tg(t)e^x - x = t\dot{x} - t\, cosh\, te^x - x$, is a first integral of (19). Also, $D_t\big(G\big(t,\omega(t,x,\dot{x})\big)\big) = D_t(t\dot{x} - t\, cosh\, te^x - x) = 0$, is a conserved form of (19).
The function $\mu(t,x,\dot{x}) = I_{\dot{x}}(t,x,\dot{x}) = t$, is an integrating factor of (19).
Therefore, we reduce the order of the equation $\ddot{x} = (sinh\, t + cosht/t + cosh\, t\dot{x})e^x$, to the equation $t\dot{x} - t\, cosh\, te^x - x = 0$. This equation dose not have Lie point symmetries.

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In this paper, we calculated an integrating factor, first integral and reduce the order the non-Linear second-order ODEs $\ddot{x} = (f(t,x) + g(t,x)\dot{x})e^x$, through λ-symmetry method. Moreover, we computed an integrating factor, first integral and reduce the order for particular cases of this equation that are $\ddot{x} = (f(t) + g(t)\dot{x})e^x$ and $\ddot{x} = (f(x) + g(x)\,\dot{x})e^x$.

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