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Majors About the New Symmetry of a Nonlinear Acoustics Model

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ABSTRACT

The Lie symmetry group of the Zabolotskaya-Khokhlov equation has been seriously studied, earlier than this. Eventually, it has been endowed with a general model by N.J.C. Ndogmo, in 2008. This research devoted to introducing the algebra, the group, and the reductions of a new symmetry which is an exception of that model.

Keyword: Zabolotskaya- Khokhlov equation (ZK), Symmetry algebra, Lie point symmetry group, Optimal system of subalgebras, Reduction of equation.

INTRODUCTION

The Zabolotskaya-Khokhlov equation in three dimensions, is a nonlinear partial differential equation in the form

$$\Delta(t,x,y,u) = u_{xt} - (u_x)^2 - uu_{xx} - u_{yy} = 0 \tag{1}$$

According to [1-3], this equation is derived from the incompressible Navier- Stokes equation and it is an acoustical mathematical model which is used in order to justify the phenomenon of sound waves propagation. N.J.C. Ndogmo in [4] addition to completing the work in [5-9], in the line of introducing Lie point symmetry group of the ZK equation, also finding its algebra and reductions, has introduced the general form of the basic elements of the symmetry algebra as

$$\begin{aligned} v_0 &= 2x\partial_x + y\partial_y + 2u\partial_u \\ v_g &= g\partial_x - g'\partial_u \\ v_h &= \frac{1}{2}yh'\partial_x + h\partial_y - \frac{1}{2}yh''\partial_u \\ v_f &= f\partial_t + \frac{1}{6}(2xf' + y^2f'')\partial_x + \frac{2}{3}yf'\partial_y \\ &\quad + \frac{1}{6}(-4uf' - 2xf'' - y^2f''')\partial_u \end{aligned} \tag{2}$$

where f, g, h are arbitrary C^∞ functions on the time variable t; dened on some open subset of $\mathbb{R}$. It is noteworthy now that, in this article has been achieved a new symmetry algebra for the ZK equation, which its base does not match with (Equa 2), and after introducing resulted Lie point symmetry group, has been obtained reductions of the equation by the one-dimensional optimal system of the new symmetry subalgebras.

THE NEW SYMMETRY GROUPS

Infinitesimal symmetries

Give for the ZK equation, the infinitesimal generators of the corresponding Lie algebra of a symmetry group of it, in the form

$$
\begin{aligned}
&v=\xi_1\partial_t+\xi_2\partial_x+\xi_3\partial_y+\phi\partial_u\\
&\xi_3+\phi=0
\end{aligned}
$$

where ξ_1,ξ_2,ξ_3,ϕ *are* C^∞ functions *of* t,x,y,u. The implementation of Maple's commands results the following linearly independent vector fields 1

$$
\begin{aligned}
v_1&=-\frac{1}{2}(y+t^2)\partial_x-t\partial_y+t\partial_u\\
v_2&=-t\partial_x-\partial_y+\partial_u\\
v_3&=\partial_x\\
v_4&=\partial_t
\end{aligned}
\tag{3}
$$

which their second prolongations are according to

$$
\begin{aligned}
pr^{(2)}v_1=&-\frac{1}{2}(y+t^2)\partial_x-t\partial_y+t\partial_u\\
&+(1+tu_x+u_y)\partial_{u_t}+\frac{1}{2}u_x\partial_{u_y}\\
&+(u_x+2tu_{tx}+2u_{ty})\partial_{u_{tt}}\\
&+(tu_{xx}+u_{xy})\partial_{u_{tx}}\\
&+\left(\frac{1}{2}u_{tx}+tu_{xy}+u_{yy}\right)\partial_{u_{ty}}\\
&+\frac{1}{2}u_{xx}\partial_{u_{xy}}+u_{xy}\partial_{u_{yy}}\\
pr^{(2)}v_2=&-t\partial_x-\partial_y+\partial_u+u_x\partial_{u_t}+2u_{tx}\partial_{u_{tt}}\\
&+u_{xx}\partial_{u_{tx}}+u_{xy}\partial_{u_{ty}}\\
pr^{(2)}v_3=&\ \partial_x\\
pr^{(2)}v_4=&\ \partial_t
\end{aligned}
$$

and they satisfy that $pr^{(2)}v_i(\Delta)=0$ with $(i=1,2,3,4)$. Thus, every $v_i(i=1,2,3,4)$ can be regarded as an innitesimal symmetry. On the other hand, $g=\langle v_1,v_2,v_3,v_4\rangle$ is a Lie algebra corresponding to a connected symmetry group of the ZK equation. That, here, it is called the new symmetry algebra.

THE LIE POINT SYMMETRY GROUP

If G is supposed as the Lie point symmetries which is produced by one parameter transformations $\exp(\varepsilon v_i)$ $(i=1234)$, $v_i\in g$, $\varepsilon\in\mathbb{R}$ then G is the same new symmetry for the ZK equation.
Since the ZK equation has involved three independent variables and one dependent variable; it can be accounted as the total space $E\simeq\mathbb{R}^{3+1}$ with coordinates t, x, y, u. So, it has Lie point transformations as

$$
\begin{aligned}
\bar{t}&=t+\zeta_1(t,x,y,u)+\mathfrak{O}(2)\\
\bar{x}&=x+\zeta_2(t,x,y,u)+\mathfrak{O}(2)\\
\bar{y}&=y+\zeta_3(t,x,y,u)+\mathfrak{O}(2)\\
\bar{u}&=u+\zeta_4(t,x,y,u)+\mathfrak{O}(2)
\end{aligned}
\tag{4}
$$

Where $\zeta_1,\zeta_2,\zeta_3,\zeta_4$ are C^∞ functions. Recall that for every $w=(t,x,y,u)$ belonging to the total Euclidean space $E\simeq\mathbb{R}^{3+1}$, $v_i\in g$ and $\varepsilon\in\mathbb{R}$, the local flow $\exp(\varepsilon v_i)\,(w)$ is defined by

$$
\partial_\varepsilon\exp(\varepsilon v_i)\,(w)=v_i|_{\exp(\varepsilon v_i)(w)}
$$

$$
\exp(\varepsilon v_i)\,(w)|_{\varepsilon=0}=w
$$

where $\partial_\varepsilon=\frac{d}{d\varepsilon}$.
Hence, the behaviour of one parameter transformations $\exp(\varepsilon v_i):E\to E$ namely

$$
\exp(\varepsilon v_i):(t,x,y,u)\mapsto(\tilde{t},\tilde{x},\tilde{y},\tilde{u}),(i=1,2,3,4),\in\mathbb{R}
\tag{5}
$$

is obtained as

Table 1: The one parameter subgroups actions

	$\tilde{t}$	$\tilde{x}$	$\tilde{y}$	$\tilde{u}$
$\exp(\varepsilon v)$	t	$-\frac{1}{4}\varepsilon^2 t-\frac{1}{2}\varepsilon t^2+x-\frac{1}{2}\varepsilon y$	$-\varepsilon t+y$	$\varepsilon t+u$
$\exp(\varepsilon v)$	t	$-\varepsilon t+x$	$y-\varepsilon$	$u+\varepsilon$
$\exp(\varepsilon v)$	t	$x+\varepsilon$	y	u
$\exp(\varepsilon v)$	$t+\varepsilon$	x	y	u

Because of the exponential map exp: $g \to G$ is the yield of replacement $\varepsilon = 1$ in the local flow; the above table deduces that

Theorem 21 *the exponential map of the new symmetry group is followed as*

$$\begin{aligned}
&\exp v_1: w \mapsto \left(t, \tfrac{1}{4}t - \tfrac{1}{2}t^2 + x - \tfrac{1}{2}y, -t + y, t + u\right)\\
&\exp v_2 : w \mapsto (t, -t + x, y - 1, u + 1)\\
&\exp v_3 : w \mapsto (t, x + 1, y, u)\\
&\exp v_4 : w \mapsto (t + 1, x, y, u)
\end{aligned} \tag{6}$$

Note that for every g belonging to G, there are elements $v_1, \cdots, v_k (k = 1,2,3,4)$ of g so $g = \exp v_1 \cdots \exp v_k$. On the other, the action of G on E is produced by combination of the one parameter subgroups actions. Therefore, the use of the Table 1 deduces that

Theorem 22 *Whenever $u = F(t, x, y)$ is a solution of the ZK equation; also, are*

$$\begin{aligned}
&u = F\left(t, \frac{1}{4}2t - \frac{1}{2}t^2 + x - \frac{1}{2}y, -t + y\right) + t\\
&u = F(t, -t + x, y -) + \varepsilon\\
&u = F(t, x+, y)\\
&u = F(t+, x, y)
\end{aligned} \tag{7}$$

Where, ε is an arbitrary parameter.

Getting $V = \alpha_1 v_1 + \cdots + \alpha_4 v_4 (\alpha_1, \cdots, \alpha_4 \in \mathbb{R})$ and applying the Maple for calculation of $\exp(\varepsilon V)$ results that

Corollary 23 *whenever $u = F(t, x, y)$ is a solution of the ZK equation; also is*

$$\begin{aligned}
u = F(t + c_1\varepsilon, x + (c_2 + c_3 t + c_4 t^2 + c_5 y)\varepsilon + (c_6 \\
+ c_7 t)\varepsilon^2 + c_8\varepsilon^3, y + (c_7 + c_9 t)\varepsilon \\
+ c_{10}\varepsilon^2) + (c_{11} + c_{12} t)\varepsilon - c_7\varepsilon^2)
\end{aligned} \tag{8}$$

where ε, $c_i (i = 1,2, \cdots, 12)$ are arbitrary numbers.

For action of the prolonged group $G^{(n)}$ on the jet space $J^n E^2 (n = 0,1,2,3, \cdots)$ give i_n as the number of functionally independent differential invariants of order n or less, similarly s_n as the maximum dimension of orbits. By means of Maple's computing, for every $n(n = 0,1,2,3, \cdot\cdot$ earns

$$s_n = \dim J^n E - i_n = 3 + \binom{n+3}{n} - i_n = 1 \tag{9}$$

Consequently, declare that

Theorem 24 *The order of stabilization and the stable orbit dimension of the new symmetry group of the ZK equations are equal to* 0 *and 1, respectively.*

Therefore, according to [10], $\dim G = 4$ indicates that

Corollary 25 *The action of the new symmetry group of the ZK equation is not locally effectively.*

STRUCTURE OF THE NEW SYMMETRY ALGEBRA

With considering to Lie brackets $[v_i, v_j] (i, j = 1,2,3,4)$, the commutator table of Lie algebra g is gathered as

Table 2: The commutation relations

$[\,,]$	v_1	v_2	v_3	v_4
v_1	0	$-2v_3$	0	$-v_2$
v_2	$-v_3$	0	0	v_3
v_3	0	0	0	0
$v4$	v_2	$-v_3$	0	0

Moreover, this algebra is solvable and nilpotent; because g's the lower and upper central series are respectively as

$$g \supset \langle 2v_2, v_3 \rangle \supset \langle v_3 \rangle \supset \{0\},$$
$$\langle v_3 \rangle \subset \langle v_2, v_3 \rangle \subset g$$

But, g is not semisimple because its killing form is generated as

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

We note that $J^0 E = E$ and $G^{(0)} = G$. Also, g is not Abelian. Nonetheless, it has Abelian subalgebras such as

$$\begin{aligned}
&\langle c_1 v_1 + c_2 v_2 + c_3 v_3 + c_4 v_4 \rangle\\
&\langle c_1 v_1 + c_2 v_2 + c_3 v_3, v_3 \rangle\\
&\langle c_3 v_3 + c_4 v_4, v_4 \rangle
\end{aligned}$$

where c_1, c_2, c_3, c_4 *are* arbitrary constants. which are all subalgebras in one and two dimensions of it. It is worth mentioning that $\langle v_3 \rangle$ is g's center. Therefore, here, the one-dimensional optimal system is as $g/\langle v_3 \rangle$. On the other, with the use of the table 2 for the Lie series

$$Ad(\exp(\varepsilon v_i))v_j = v_i - [v_i, v_j] \pm \frac{\varepsilon^2}{2}[v_i, [v_i, v_j]] - \cdots \qquad (10)$$

obtains

Table 3: The adjoint actions

Ad	v_1	v_2	v_3	v_4
v_1	v_1	$v_2 - \varepsilon v_4$	$-\frac{1}{2}\varepsilon v_2 + v_3 + \frac{1}{4}\varepsilon^2 v_4$	v_4
v_2	v_1	v_2	$\frac{1}{2}\varepsilon v_1 + v_3 + \varepsilon v_4$	v_4
v_3	v_1	v_2	v_3	v_4
v_4	v_1	$\varepsilon v_1 + v_2$	$-\frac{1}{2}\varepsilon^2 v_1 - \varepsilon v_2 + v_3$	v_4

with the (i,j)-th entry indicating $\mathrm{Ad}(\exp(\varepsilon v_i))v_j$ $(i,j = 1,2,3,4)$. It leads us to the next theorem's proof

Theorem 31 *The one*-dimensional *optimal system of new symmetric subalgebras of the ZK equation is conformity with*

$\langle v_3\rangle$, $\langle v_1 + c_3 v_3\rangle$, $\langle v_2 + c_3 v_3\rangle$, $\langle v_4 + c_3 v_3\rangle$, $\langle c_1 v_1 + c_2 v_2 + v_3\rangle$, $\langle c_1 v_1 + v_3 + c_4 v_4\rangle$, $\langle c_2 v_2 + v_3 + c_4 v_4\rangle$, $\langle c_1 v_1 + c_2 v_2 + v_3 + c_4 v_4\rangle$.

where c_1, c_2, c_3, c_4 *are* nonzero constants.

REDUCTIONS OF THE ZK EQUATION BY THE ONE-DIMENSIONAL OPTIMAL SYSTEM

With founding and solving the characteristic equations of generators of the one-dimensional optimal system of subalgebras, the next table can be obtained

Table 4: The one-dimensional optimal system's invariants

	The invariants up to the second order
v_3	$t, y, u, u_t, u_x, u_y, u_{tt}, u_{tx}, u_{ty}, u_{xx}, u_{xy}, u_{yy}$
$v_1 + v_3$	$t, -x - \frac{1}{t}y + \frac{1}{4t}y^2 + \frac{1}{2}ty, u + y + t^2 - 2, u_t, u_x, u_y, u_{tt}, u_{tx}, u_{ty}, u_{xx}, u_{xy}, u_{yy}$
$v_2 + v_3$	$t, y - \frac{x}{t-1}, u + \frac{x}{t-1}, u_t, u_x, u_y, u_{tt}, u_{tx}, u_{ty}, u_{xx}, u_{xy}, u_{yy}$
$v_3 + v_4$	$x - t, y, u, u_t, u_x, u_y, u_{tt}, u_{tx}, u_{ty}, u_{xx}, u_{xy}, u_{yy}$
$v_1 + v_2 + v_3$	$t, -4x + \frac{4y(t-1)+y^2+2t^2y}{t+1}, u + y + t^2 + 2t, u_t, u_x, u_y, u_{tt}, u_{tx}, u_{ty}, u_{xx}, u_{xy}, u_{yy}$
$v_2 + v_3 + v_4$	$t, \frac{1}{2}t^2 - t + x, y + t, u - t, u_t, u_x, u_y, u_{tt}, u_{tx}, u_{ty}, u_{xx}, u_{xy}, u_{yy}$
$v_1 + v_3 + v_4$	$t, \frac{1}{2}t^2 + y, \frac{1}{3}t^3 - t + \frac{1}{2}ty + x, -\frac{1}{2}t^2 + u, u_t, u_x, u_y, u_{tt}, u_{tx}, u_{ty}, u_{xx}, u_{xy}, u_{yy}$
$v_1 + v_2 + v_3 + v_4$	$t, y + \frac{1}{2}t^2 + t, x + \frac{1}{12}t^3 + \frac{1}{4}t^2 - t + \frac{1}{2}t(y + \frac{1}{2}t^2 + t), u - \frac{1}{2}t^2 - t, u_t, u_x, u_y, u_{tt}, u_{tx}, u_{ty}, u_{xx}, u_{xy}, u_{yy}$

Every of this table rows contains three ordinary invariants that are exerted to formulate relations which substituting of them in the ZK equation reduces the number of its independent variables as (refer to[10])

i) The reduction by $\langle v_3\rangle$ is done by the formula

$$u(t,x,y) = s(w,z)\ , t = w, y = z \qquad (11)$$

and the reduced equation is

$$s_{zz} = 0 \qquad (12)$$

That, its corresponding invariant solution for Equ (1) is $u = f(t)y + g(t)$, where f, g are arbitrary C^∞ functions.

ii) The reduction by $\langle v_1 + v_3\rangle$ is done by the formula

$$u(t,x,y) = s(w,z) - w^2 - y + 2$$
$$t = w$$
$$x = -z - \frac{1}{w}y + \frac{1}{4w}y^2 + \frac{1}{2}wy \qquad (13)$$

and the reduced equation is

$$ws_{wz} + w(-w^2 - y + s + 3)s_{zz} + ws_z^2 + s_z = 0 \qquad (14)$$

iii) the reduction by $\langle v_2 + v_3 \rangle$ is done by the formula

$$u(t,x,y) = s(w,z) - \frac{x}{w-1}$$
$$t = w$$
$$y = z + \frac{x}{w-1} \tag{15}$$

and the reduced equation is

$$(w-1)s_{wz} + (s + (w-1)^2)s_{zz} + s_z^2 + s_z = 0 \tag{16}$$

iv) The reduction by $\langle v_3 + v_4 \rangle$ is done by the formula

$$u(t,x,y) = s(w,z) \quad , \quad t = x - w \quad , \quad y = z \tag{17}$$

and the reduced equation is

$$(s+1)s_{ww} + s_w^2 + s_{zz} = 0 \tag{18}$$

v) the reduction by $\langle v_1 + v_2 + v_3 \rangle$ is done by the formula

$$u(t,x,y) = s(w,z) - y - w^2 - 2w + 2$$
$$t = w$$
$$x = -\frac{1}{4}z + \frac{y(w-1)}{w+1} + \frac{y^2}{4(w+1)} + \frac{w^2 y}{2(w+1)} \tag{19}$$

and the reduced equation is

$$2(w+1)^2 s_{wz} + (-6w^2(w+2)^2 + 8s(w+1)^2 + 24)s_{zz} + 8(w+1)^2 s_z^2 + (w+1)s_z = 0 \tag{20}$$

vi) the reduction by $\langle v_2 + v_3 + v_4 \rangle$ is done by the formula

$$u(t,x,y) = s(w,z) + t$$
$$x = w + t - \frac{1}{2}t^2$$
$$y = z - t \tag{21}$$

and the reduced equation is

$$s_{wz} - s_{zz} - (s+1)s_{ww} - s_w^2 = 0 \tag{22}$$

vii) the reduction by $\langle v_1 + v_3 + v_4 \rangle$ is done by the formula

$$u(t,x,y) = s(w,z) + \frac{1}{2}t^2$$
$$x = z - \frac{1}{2}tw - \frac{1}{12}t^3 + t$$
$$y = w - \frac{1}{2}t^2 \tag{23}$$

and the reduced equation is

$$ts_{wz} - s_{ww} - \left(\frac{1}{4}t^2 + s + 1\right)s_{zz} - s_z^2 = 0 \tag{24}$$

viii) the reduction by $\langle v_1 + v_2 + v_3 + v_4 \rangle$ is done by the formula

$$u(t,x,y) = s(w,z) + \frac{1}{2}t^2 + t$$
$$x = z - \frac{1}{12}t^3 - \frac{1}{4}t^2 + t - \frac{1}{2}tw$$
$$y = w - \frac{1}{2}t^2 - t \tag{25}$$

and the reduced equation is

$$(t+1)s_{wz} - s_{ww} - \left(\frac{1}{4}t^2 + \frac{1}{2}t + s + 1\right)s_{zz} - s_z^2 = 0 \tag{26}$$

CONCLUSION

In this article, attaching options in the Maple's commands led us to break the former expressed generalization about the ZK equation's Lie symmetries, and this attainment paves the way for more detailed studies of this equation.

REFERENCES

[1] E. J. S. P. A. Zabolotskaya, "Quasi-plane waves, in the nonlinear acoustics of confined beams," vol. 15, pp. 35-40, 1969.

[2] A. P. Sarvazyan, O. V. Rudenko, S. D. Swanson, J. B. Fowlkes, S. Y. J. U. i. m. Emelianov, and biology, "Shear wave elasticity imaging: a new

ultrasonic technology of medical diagnostics," vol. 24, no. 9, pp. 1419-1435, 1998.

[3] N. Bakhvalov, I. A. M. Zhileĭkin, I. A. k. M. Zhileĭkin, Y. Zhileikin, and E. Zabolotskaya, *Nonlinear theory of sound beams*. American Institute of Physics, 1987.

[4] J. J. N. D. Ndogmo, "Symmetry properties of a nonlinear acoustics model," vol. 55, no. 1, pp. 151-167, 2009.

[5] A. R. Chowdhury, M. J. J. o. P. A. M. Nasker, and General, "Towards the conservation laws and Lie symmetries for the Khokhlov-Zabolotskaya equation in three dimensions," vol. 19, no. 10, p. 1775, 1986.

[6] F. J. J. o. P. A. M. Schwarz, Nuclear and General, "Symmetries of the Khokhlov-Zabolotskaya equation," vol. 20, no. 6, pp. 1613-1614, 1987.

[7] W. Hereman, W.-H. Steeb, N. J. J. o. P. A. M. Euler, and General, "Comment on'Towards the conservation laws and Lie symmetries for the Khokhlov-Zabolotskaya equation in three dimensions'," vol. 25, no. 8, p. 2417, 1992.

[8] J.-F. Zhang, Y.-J. Zhu, and J. J. C. i. T. P. Lin, "Similarity reductions for the Khokhlov–Zabolotskaya equation," vol. 24, no. 1, p. 69, 1995.

[9] P. A. Clarkson and M. D. J. J. o. M. P. Kruskal, "New similarity reductions of the Boussinesq equation," vol. 30, no. 10, pp. 2201-2213, 1989.

[10] P. J. Olver, *Equivalence, invariants and symmetry*. Cambridge University Press, 1995.